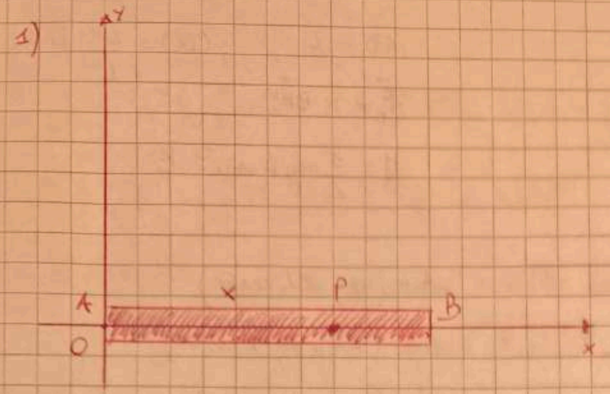
1)

24/04/2025



$\overline{AB} = L$

$\rho(P) = \frac{2m}{L^2}\,|AP|$

Individuiamo la posizione del baricentro G:

$$M = \int_0^L \rho(x)\,dx = \frac{2m}{L^2}\int_0^L x\,dx = \frac{2m}{L^2}\left[\frac{x^2}{2}\right]_0^L = \frac{2m}{L^2}\cdot\frac{L^2}{2} = m \quad \Rightarrow \quad M = m$$

$$x_G = \frac{1}{m}\int_0^L \rho\, x\,dx = \frac{1}{m}\,\frac{2m}{L^2}\int_0^L x^2\,dx = \frac{2}{L^2}\left[\frac{x^3}{3}\right]_0^L = \frac{2}{L^2}\cdot\frac{L^3}{3} = \frac{2}{3}L$$

Ne viene che:

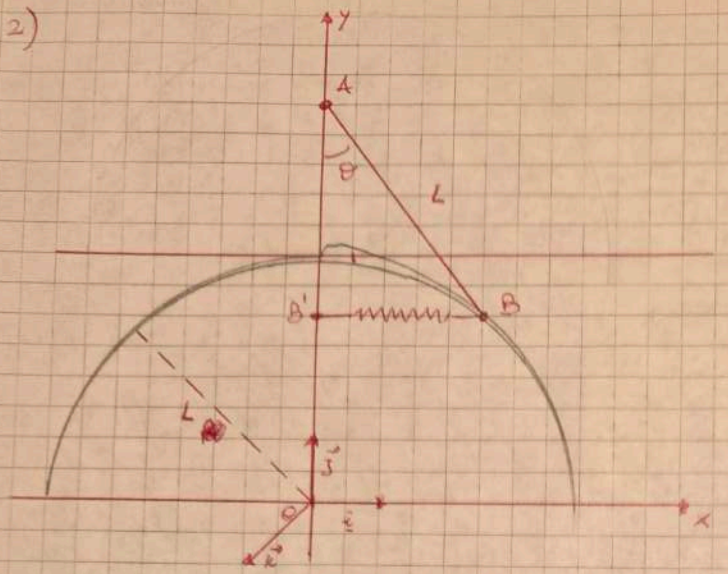
$$I_G = \int_0^L \rho(x)\left(x - \frac{2}{3}L\right)^2 dx = \frac{2m}{L^2}\int_0^L x\left(x^2 + \frac{4}{9}L^2 - \frac{4}{3}Lx\right)dx =$$

$$= \frac{2m}{L^2}\int_0^L \left(x^3 + \frac{4}{9}L^2 x - \frac{4}{3}Lx^2\right)dx =$$

$$= \frac{2m}{L^2}\left[\frac{x^4}{4} + \frac{4}{9}L^2\frac{x^2}{2} - \frac{4}{3}L\frac{x^3}{3}\right]_0^L = \frac{2m}{L^2}\left(\frac{L^4}{4} + \frac{2}{9}L^4 - \frac{4}{9}L^4\right) =$$

$$= \frac{2m}{L^2}\left(\frac{9+8-16}{36}\right)L^4 = \frac{1}{18}mL^2$$

2)



$\overline{AB} = L$, $\rho(Q) = \frac{2m}{L^2}|AQ|$

$\vec{F}_e = k\,\vec{BB'}$

$\underline{M} = \frac{1}{6} mgL \sin\vartheta\, \underline{k}$

$A = (0;\ 2L\cos\vartheta)$

$B = (L\sin\vartheta;\ L\cos\vartheta)$

$B' = (0;\ L\cos\vartheta)$

$G = \left(\frac{2}{3}L\sin\vartheta;\ 2L\cos\vartheta - \frac{2}{3}L\cos\vartheta\right)$

$= \left(\frac{2}{3}L\sin\vartheta;\ \frac{4}{3}L\cos\vartheta\right)$

L'energia cinetica si ottiene applicando il teorema di König:

$$T = \frac{1}{2} m v_G^2 + \frac{1}{2} I_G \omega^2$$

$$\underline{v}_G = \left(\frac{2}{3}L\dot{\vartheta}\cos\vartheta;\ -\frac{4}{3}L\dot{\vartheta}\sin\vartheta\right) \Rightarrow v_G^2 = \frac{4}{9}L^2\dot{\vartheta}^2\cos^2\vartheta + \frac{16}{9}L^2\dot{\vartheta}^2\sin^2\vartheta =$$

$$= \frac{4}{3}L^2\dot{\vartheta}^2(\cos^2\vartheta + 4\sin^2\vartheta)$$

Dall'esercizio precedente: $I_G = \frac{1}{18} mL^2$. Quindi:

$$T = \frac{1}{2} m \cdot \frac{4}{9} L^2\dot{\vartheta}^2(\cos^2\vartheta + 4\sin^2\vartheta) + \frac{1}{2}\cdot\frac{1}{18} mL^2\dot{\vartheta}^2 =$$

$$= \frac{2}{9} mL^2\dot{\vartheta}^2(\cos^2\vartheta + 4\sin^2\vartheta) + \frac{1}{36} mL^2\dot{\vartheta}^2 =$$

$$= \frac{2}{9} mL^2\dot{\vartheta}^2 + \frac{2}{3} mL^2\dot{\vartheta}^2\sin^2\vartheta + \frac{1}{36} mL^2\dot{\vartheta}^2 = \frac{1}{4} mL^2\dot{\vartheta}^2 + \frac{2}{3} mL^2\dot{\vartheta}^2\sin^2\vartheta$$

Passiamo al calcolo del potenziale. Il lavoro compiuto dalla coppia $\underline{M}$ è un differenziale esatto, poiché:

$$\delta L = \underline{M}\cdot\delta\vartheta\,\underline{k} = \left(\frac{1}{6} mgL\sin\vartheta\,\underline{k}\right)\cdot\delta\vartheta\,\underline{k} = \frac{1}{6} mgL\sin\vartheta\,\delta\vartheta = \delta\left(-\frac{1}{6} mgL\cos\vartheta\right)$$

Pertanto esiste il potenziale: $U_M = -\frac{1}{6} mgL\cos\vartheta$.

Il potenziale totale è dato perciò da:

$$U = U_P + U_e + U_M =$$

$$= -mg\,y_G - \frac{1}{2}k\,|BB'|^2 + U_M = -\frac{4}{3}mgL\cos\vartheta - \frac{1}{2}kL^2\sin^2\vartheta - \frac{1}{6}mgL\cos\vartheta =$$

$$= -\frac{3}{2}mgL\cos\vartheta - \frac{1}{2}kL^2\sin^2\vartheta$$

La Lagrangiana del sistema risulta essere:

$$\mathcal{L} = T + U = \frac{1}{4}mL^2\dot\vartheta^2 + \frac{2}{3}mL^2\dot\vartheta^2\sin^2\vartheta - \frac{3}{2}mgL\cos\vartheta - \frac{1}{2}kL^2\sin^2\vartheta$$

L'equazione di Lagrange per $q_1 = \vartheta$ è:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial\dot\vartheta}\right) - \frac{\partial \mathcal{L}}{\partial\vartheta} = 0$$

$$\frac{\partial \mathcal{L}}{\partial\dot\vartheta} = \frac{1}{2}mL^2\dot\vartheta + \frac{4}{3}mL^2\dot\vartheta\sin^2\vartheta \Rightarrow$$

$$\left.\begin{aligned}
\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial\dot\vartheta}\right) &= \frac{1}{2}mL^2\ddot\vartheta + \frac{4}{3}mL^2\ddot\vartheta\sin^2\vartheta + \frac{4}{3}mL^2\dot\vartheta\cdot 2\sin\vartheta\cdot\cos\vartheta\cdot\dot\vartheta = \\
&= \frac{1}{2}mL^2\ddot\vartheta + \frac{4}{3}mL^2\ddot\vartheta\sin^2\vartheta + \frac{8}{3}mL^2\dot\vartheta^2\sin\vartheta\cos\vartheta \\
\frac{\partial \mathcal{L}}{\partial\vartheta} &= \frac{4}{3}mL^2\dot\vartheta^2\sin\vartheta\cos\vartheta + \frac{3}{2}mgL\sin\vartheta - kL^2\sin\vartheta\cos\vartheta
\end{aligned}\right\} \Rightarrow$$

$$\frac{1}{2}mL^2\ddot\vartheta + \frac{4}{3}mL^2\ddot\vartheta\sin^2\vartheta + \frac{8}{3}mL^2\dot\vartheta^2\sin\vartheta\cos\vartheta - \frac{4}{3}mL^2\dot\vartheta^2\sin\vartheta\cos\vartheta - \frac{3}{2}mgL\sin\vartheta +$$
$$+ kL^2\sin\vartheta\cos\vartheta = 0 \Rightarrow$$

$$mL^2\ddot\vartheta\left(\frac{1}{2} + \frac{4}{3}\sin^2\vartheta\right) + \frac{4}{3}mL^2\dot\vartheta^2\sin\vartheta\cos\vartheta - \frac{3}{2}mgL\sin\vartheta + kL^2\sin\vartheta\cos\vartheta = 0$$